

# CONTINUOUS RATIONAL FUNCTIONS ON REAL AND $p$ -ADIC VARIETIES

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A *rational function* on  $\mathbb{R}^n$  is a quotient of two polynomials

$$f(x_1, \dots, x_n) := \frac{p(x_1, \dots, x_n)}{q(x_1, \dots, x_n)}.$$

Strictly speaking, a rational function  $f$  is not really a function on  $\mathbb{R}^n$  in general since it is defined only on the dense open set where  $q \neq 0$ . Nonetheless, even if  $q$  vanishes at some points of  $\mathbb{R}^n$ , it can happen that there is an everywhere defined continuous function  $f^c$  that agrees with  $f$  at all points where  $f$  is defined. Such an  $f^c$  is unique. For this reason it is customary to identify  $f$  with  $f^c$  and call  $f$  itself a *continuous rational function* on  $\mathbb{R}^n$ . For instance,

$$\frac{p(x_1, \dots, x_n)}{x_1^{2m} + \dots + x_n^{2m}}$$

is a continuous rational function on  $\mathbb{R}^n$  if every monomial in  $p$  has degree  $> 2m$ .

The above definition of continuous rational functions makes sense on any real algebraic variety  $X$ , as long as the open set where  $f$  is defined is dense in  $X$  in the Euclidean topology. This condition always holds on smooth varieties, and, more generally, if the singular set is contained in the Euclidean closure of the smooth locus.

The aim of this note is to consider four basic problems on continuous rational functions.

**Question 1.** Let  $X$  be a real algebraic variety and  $Z$  a closed subvariety.

- (1) Let  $f$  be a continuous rational function defined on  $X$ . Is the restriction  $f|_Z$  a rational function on  $Z$ ?
- (2) Let  $g$  be a continuous rational function defined on  $Z$ . Can one extend it to a continuous rational function on  $X$ ?
- (3) Assume that  $X \setminus Z$  is Zariski dense in  $X$ . Is  $f$  uniquely determined by the restriction  $f|_{X \setminus Z}$ ?
- (4) Which systems of linear equations

$$\sum_j f_{ij} \cdot y_j = g_i \quad i = 1, \dots, m$$

have continuous rational solutions where the  $g_i, f_{ij}$  are polynomials (or rational functions) on  $X$ .

Section 1 contains a series of counter examples to Questions 1.1–3. Proposition 7 shows that the answer to Question 1.1 is always yes if  $X$  is smooth. In Section 2 this leads to the introduction of the notion of *hereditarily rational functions*, also studied in [FHMM11]. (Continuous rational maps between smooth algebraic varieties were investigated, from a perspective oriented more at topological aspects, in [Kuch09].) These have the good properties that one would expect based on the smooth case.

In Theorem 9 this concept is used to give a complete answer to Question 1.2. We do not have a full answer to Question 1.4.

All these results extend to  $p$ -adic and other topological fields, see Section 3.

### 1. EXAMPLES

The following example shows that the answer to Questions 1.1–2 is not always positive.

**Example 2.** Consider the surface  $S$  and the rational function  $f$  given by

$$S := (x^3 - (1 + z^2)y^3 = 0) \subset \mathbb{R}^3 \quad \text{and} \quad f(x, y, z) := \frac{x}{y}.$$

We claim that

- (1)  $S$  is a real analytic submanifold of  $\mathbb{R}^3$ ,
- (2)  $f$  is defined away from the  $z$ -axis,
- (3)  $f$  extends to a real analytic function  $f^c$  on  $S$ , yet
- (4) the restriction of  $f^c$  to the  $z$ -axis is not rational and
- (5)  $f$  can not be extended to a continuous rational function on  $\mathbb{R}^3$ .

Proof. Note that

$$x^3 - (1 + z^2)y^3 = (x - (1 + z^2)^{1/3}y)(x^2 + (1 + z^2)^{1/3}xy + (1 + z^2)^{2/3}y^2).$$

The first factor defines  $S$  as a real analytic submanifold of  $\mathbb{R}^3$ . The second factor only vanishes along the  $z$ -axis which is contained in  $S$ . Therefore  $x - y\sqrt[3]{1 + z^2}$  vanishes on  $S$ , hence

$$f^c|_S = \frac{x}{y}|_S = \sqrt[3]{1 + z^2}|_S \quad \text{and so} \quad f(0, 0, z) = \sqrt[3]{1 + z^2}.$$

Assume finally that  $F$  is a continuous rational function on  $\mathbb{R}^3$  whose restriction to  $S$  is  $f$ . Then  $F$  and  $f$  have the same restrictions to the  $z$ -axis. We show below that the restriction of any continuous rational function  $F$  defined on  $\mathbb{R}^3$  to the  $z$ -axis is a rational function. Thus  $F|_{z\text{-axis}}$  does not equal  $\sqrt[3]{1 + z^2}$ , a contradiction.

To see the claim, write  $F = p(x, y, z)/q(x, y, z)$  where  $p, q$  are polynomials. We may assume that they are relatively prime. Since  $F$  is continuous everywhere,  $x$  can not divide  $q$ . Hence  $F|_{(x=0)} = p(0, y, z)/q(0, y, z)$ . By canceling common factors, we can write this as  $F|_{(x=0)} = p_1(y, z)/q_1(y, z)$  where  $p_1, q_1$  are relatively prime polynomials. As before,  $y$  can not divide  $q_1$ , hence  $F|_{z\text{-axis}} = p_1(0, z)/q_1(0, z)$  is a rational function.

(Note that we seemingly have not used the continuity of  $F$ : for any rational function  $f(x, y, z)$  the above procedure defines a rational function on the  $z$ -axis. However, if we use  $x, y$  in reverse order, we could get a different rational function. This happens, for instance, for  $f = x^2z/(x^2 + y^2)$ . Here  $(f|_{(x=0)})|_{z\text{-axis}} = 0$  and  $(f|_{(y=0)})|_{z\text{-axis}} = z$ .)

In the above example, the problems arise since  $S$  is not normal. However, the key properties (2.4–5) can also be realized on a normal hypersurface.

**Example 3.** Consider

$$X := ((x^3 - (1 + t^2)y^3)^2 + z^6 + y^7 = 0) \subset \mathbb{R}^4 \quad \text{and} \quad f(x, y, z, t) := \frac{x}{y}.$$

We easily see that the singular locus is the  $t$ -axis.  $X$  is normal since a hypersurface in a smooth variety is normal iff its singular set has codimension  $\geq 2$ ; see for instance [Ser65, Chap.III.C, Prop.9] or [Eis04, Thm.11.2].

Let us blow up the  $t$ -axis. There is one relevant chart, where  $x_1 = x/y, y_1 = y, z_1 = z/y$ . We get the smooth 3-fold

$$X' := ((x_1^3 - (1 + t^2))^2 + z_1^6 + y_1 = 0) \subset \mathbb{R}^4.$$

Each point  $(0, 0, 0, t)$  has only 1 preimage in  $X'$ , given by  $(\sqrt[3]{1+t^2}, 0, 0, t)$  and the projection  $\pi : X' \rightarrow X$  is a homeomorphism. Thus  $X$  is a topological manifold, but it is not a differentiable submanifold of  $\mathbb{R}^4$ .

Since  $f \circ \pi = x_1$  is a regular function, we conclude that  $f \circ \pi$  extends to a continuous (even regular) function  $(f \circ \pi)^c$  on  $X'$ . Since  $\pi$  is a homeomorphism, we get the continuous function  $f^c := (f \circ \pi)^c \circ \pi^{-1}$  on  $X$  extending  $f$ . By construction,  $f(0, 0, 0, t) = \sqrt[3]{1+t^2}$ , thus, as before,  $f$  can not be extended to a continuous rational function on  $\mathbb{R}^4$ .

For any  $m \geq 1$  we get similar examples of normal hypersurfaces and rational functions

$$X_m := ((x^3 - (1 + t^2)y^3)^2 + z_1^6 + \dots + z_m^6 + y^7 = 0) \subset \mathbb{R}^{3+m} \quad \text{and} \quad f := \frac{x}{y}.$$

Note that for all  $m$ , the singular set is still the  $t$ -axis and for  $m \gg 1$ , the  $X_m$  have rational, even terminal singularities (see [KM98] for the definitions of these singularities). In fact, we do not know any natural class of singularities (other than smooth points) where Questions 1.1–2 have a positive answer.

In order to elucidate Question 1.3, next we give an example of a continuous rational function  $f$  on  $\mathbb{R}^3$  and of an irreducible algebraic surface  $S \subset \mathbb{R}^3$  such that  $f|_S$  is zero on a Zariski dense, Zariski open subset of  $S$  yet  $f^c|_S$  is not identically zero.

**Example 4.** Consider the rational function

$$f(x, y, z) := z^2 \cdot \frac{x^2 + y^2 z^2 - y^3}{x^2 + y^2 z^2 + y^4}.$$

Its only possible discontinuities are along the  $z$ -axis. To analyze its behavior there, rewrite it as

$$f = z^2 - y(1 + y) \cdot \frac{y^2 z^2}{x^2 + y^2 z^2 + y^4}.$$

The fraction is bounded by 1, hence  $f$  extends to a continuous function  $f^c$  and  $f^c(0, 0, z) = z^2$ .

Our example is the restriction of  $f$  to the surface  $S := (x^2 + y^2 z^2 - y^3 = 0) \subset \mathbb{R}^3$ . The projection of  $S$  to the  $(y, z)$ -plane is the inside of the parabola  $(y \geq z^2) \subset \mathbb{R}_{yz}^2$ . Topologically,  $S$  has 2 parts. One is the  $z$ -axis, which is also the singular locus of  $S$ , and the other part  $S^*$  is the Euclidean closure of the smooth locus of  $S$ . The 2 parts intersect only at the origin. Thus  $S^*$  is Zariski dense but not Euclidean dense in  $S$ .

We see that  $f^c$  vanishes on  $S^*$  but not on the  $z$ -axis.

More generally, let  $g(z)$  be any rational function without real poles. Then  $g(z)f(x, y, z)$  vanishes on  $S^*$  and its restriction to the  $z$ -axis is  $z^2 g(z)$ .

(The best known example of a surface with a Zariski dense open set which is not Euclidean dense is the Whitney umbrella  $W := (x^2 = y^2 z) \subset \mathbb{R}^3$ . We can take  $W_1 := (x = y = 0)$  and  $W_2 := W$ . The Euclidean closure of  $W_2 \setminus W_1$  does not contain the “handle”  $(x = y = 0, z < 0)$ . In this case, a continuous rational function is determined by its restriction to  $W_2 \setminus W_1$ . The Euclidean closure of

$W_2 \setminus W_1$  contains the half line  $(x = y = 0, z \geq 0)$ , and a rational function on a line is determined by its restriction to any interval.)

The next example shows the two natural ways of pulling back continuous rational functions by a morphism can be different.

**Example 5** (Two pull-backs). Let  $\pi : X' \rightarrow X$  be a morphism of real varieties. Assume for simplicity that  $X, X'$  are irreducible and  $\pi$  is birational. If  $f$  is a continuous rational function on  $X$ , then one can think of the pull-back of  $f$  or of  $f^c$  to  $X'$  in at least two different ways.

First,  $f^c \circ \pi$  is the composite of two continuous maps, hence it is a continuous function. Second, one can view  $f \circ \pi$  as a rational function on  $X'$ . Let  $U' \subset X'$  be the open set where  $f \circ \pi$  is regular. Let us denote the resulting regular function by  $(f \circ \pi)^r : U' \rightarrow \mathbb{R}$ .

The following example shows that  $(f \circ \pi)^r$  and  $(f^c \circ \pi)|_{U'}$  can be different.

Take  $X = \mathbb{R}^2$  with  $f = x^3/(x^2 + y^2)$ . Note that  $f^c(0, 0) = 0$ . Blow up  $(x^3, x^2 + y^2)$  to obtain  $X' \subset \mathbb{R}^2 \times \mathbb{RP}^1$ . The first projection  $\pi : X' \rightarrow X$  is an isomorphism away from the origin and  $\pi^{-1}(0, 0) \cong \mathbb{RP}^1$ .

The first interpretation above gives a continuous function  $f^c \circ \pi$  which vanishes along  $\pi^{-1}(0, 0)$ .

The second interpretation views  $f \circ \pi$  as a rational map

$$f \circ \pi : X' \setminus \pi^{-1}(0, 0) \dashrightarrow \mathbb{RP}^1$$

which is in fact regular on  $X'$ . Thus it extends to a continuous (even regular) map  $(f \circ \pi)^c$  on  $X'$  whose restriction to  $\pi^{-1}(0, 0)$  is an isomorphism  $\pi^{-1}(0, 0) \cong \mathbb{RP}^1$ .

This confusion is possible only because  $X'_0 := \pi^{-1}(\mathbb{R}^2 \setminus (0, 0))$  is not Euclidean dense in  $X'$ . Its Euclidean closure contains only one point of  $\pi^{-1}(0, 0) \cong \mathbb{RP}^1$ . The two versions of  $f \circ \pi$  agree on  $X'_0$ , hence also on its Euclidean closure, but not everywhere.

In general, let  $\pi : X' \rightarrow X$  be a morphism of real varieties and  $f$  a rational function on  $X$  that is regular on an open set  $U \subset X$ . Let  $U' \subset X'$  be the Euclidean closure of  $\pi^{-1}(U)$ ; it is a semialgebraic subset of  $X'$ . All possible definitions of a continuous pull-back of  $f$  agree on  $U'$  but, as the above example shows, they may be different on  $X' \setminus U'$ .

Finally we turn to Question 1.4 for a single equation

$$\sum_i f_i(\mathbf{x}) \cdot y_i = g(\mathbf{x}),$$

where  $g$  and the  $f_i$  are polynomials in the variables  $\mathbf{x} = (x_1, \dots, x_n)$ . Such equations have a solution where the  $y_i$  are rational functions provided not all of the  $f_i$  are identically zero. The existence of a solution where the  $y_i$  are continuous functions is studied in [FK11] and [Kol10].

One could then hope to prove that if there is a continuous solution then there is also a continuous rational solution. [FK11, Sec.2] proves that if there is a continuous solution then there is also a continuous semialgebraic solution. The next example shows, however, that in general there is no continuous rational solution.

**Example 6.** We claim that the linear equation

$$x_1^3 x_2 \cdot y_1 + (x_1^3 - (1 + x_3^2)x_2^3) \cdot y_2 = x_1^4 \quad (6.1)$$

has a continuous semialgebraic solution but no continuous rational solution.

A continuous semialgebraic solution is given by

$$y_1 = (1 + x_3^2)^{1/3} \quad \text{and} \quad y_2 = \frac{x_1^3}{x_1^2 + (1 + x_3^2)^{1/3}x_1x_2 + (1 + x_3^2)^{2/3}x_2^2}. \quad (6.2)$$

(Note that  $x_1^2 + (1 + x_3^2)^{1/3}x_1x_2 + (1 + x_3^2)^{2/3}x_2^2 \geq \frac{1}{2}(x_1^2 + x_2^2)$ , so  $|y_2| \leq 2x_1$  and it is indeed continuous.) A solution by rational functions is  $y_1 = x_1/x_2$  and  $y_2 = 0$ .

To see that (6.1) has no continuous rational solution, restrict any solution to the semialgebraic surface  $S := (x_1 - (1 + x_3^2)^{1/3}x_2 = 0)$ . Since  $x_1^3 - (1 + x_3^2)x_2^3$  is identically zero on  $S$ , we conclude that  $y_1|_S = x_1^4/(x_1^3x_2)|_S = (x_1/x_2)|_S$ . The latter is equal to  $\sqrt[3]{1 + x_3^2}$ , thus

$$y_1|_{x_3\text{-axis}} = \sqrt[3]{1 + x_3^2}$$

which is not a rational function. As we saw in Example 2, this implies that  $y_1$  is not a rational function.

## 2. HEREDITARILY RATIONAL FUNCTIONS

First we show that Question 1.1 has a positive answer on smooth varieties.

**Proposition 7.** *Let  $X$  be a real algebraic variety and  $Z$  an irreducible subvariety that is not contained in the singular locus of  $X$ . Let  $f$  be a rational function on  $X$  with continuous extension  $f^c$ . Then there is a Zariski dense open subset  $Z^0 \subset Z$  such that  $f^c|_{Z^0}$  is a regular function.*

*Proof.* By replacing  $X$  with a suitable open subvariety, we may assume that  $X$  and  $Z$  are both smooth.

Assume first that  $Z$  has codimension 1. Then the local ring  $\mathcal{O}_{X,Z}$  is a principal ideal domain [Sha94, Sec.II.3.1]; let  $t \in \mathcal{O}_{X,Z}$  be a defining equation of  $Z$ . We can write  $f = t^m u$  where  $m \in \mathbb{Z}$  and  $u \in \mathcal{O}_{X,Z}$  is a unit. Here  $m \geq 0$  since  $f$  does not have a pole along  $Z$ , hence  $f$  is regular along a Zariski dense open subset  $Z^0 \subset Z$ . Thus  $f|_{Z^0}$  is a regular function.

If  $Z$  has higher codimension, note that  $Z$  is a local complete intersection at its smooth points [Sha94, Sec.II.3.2]. That is, there is a sequence of subvarieties  $Z \supset Z_0 \subset Z_1 \subset \cdots \subset Z_m = X_0 \subset X$  where each  $Z_i$  is a smooth hypersurface in  $Z_{i+1}$  for  $i = 0, \dots, m-1$  and  $Z_0$  (resp.  $X_0$ ) is open and dense in  $Z$  (resp.  $X$ ). We can thus restrict  $f^c$  to  $Z_{m-1}$ , then to  $Z_{m-2}$  and so on, until we get that  $f^c|_{Z_0}$  is regular. (As we noted in Example 6, for any rational function  $f$  the above procedure defines a regular function  $f|_{Z_0}$ , but it depends on the choice of the chain  $Z_1 \subset \cdots \subset Z_m$ .)  $\square$

This suggests that we should focus on those functions for which Question 1.1 has a positive answer. Then we show that for such functions Question 1.2 also has a positive answer.

**Definition 8.** Let  $X$  be a real algebraic variety and  $h$  a continuous function on  $X$ . We say that  $h$  is *hereditarily rational* if every irreducible, real subvariety  $Z \subset X$  has a Zariski dense open subvariety  $Z^0 \subset Z$  such that  $h|_{Z^0}$  is a regular function on  $Z^0$ . (A function  $h$  is *regular at*  $x \in X$  if one can write  $h = p/q$  where  $p, q$  are polynomials and  $q(x) \neq 0$ . It is called *regular* if it is regular at every point of  $X$ .)

Examples 2 and 3 show that not every continuous rational function is hereditarily rational.

If  $f$  is rational, there is a Zariski dense open set  $X^0 \subset X$  such that  $f|_{X^0}$  is regular. If  $h$  is continuous and hereditarily rational, we can repeat this process with the restriction of  $f$  to  $X \setminus X^0$ , and so on. Thus we conclude that a continuous function  $h$  is hereditarily rational iff there is a sequence of closed subvarieties  $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_m = X$  such that for  $i = 0, \dots, m$  the restriction of  $h$  to  $X_i \setminus X_{i-1}$  is regular.

If it is convenient, we can also assume that each  $X_i \setminus X_{i-1}$  is smooth of pure dimension  $i$ .

Proposition 7 says that on a smooth variety every continuous rational function is hereditarily rational.

The pull-back of a hereditarily rational function by any morphism is again a (continuous and) hereditarily rational function.

The following result shows that hereditarily rational functions constitute the right class for Question 1.2.

**Theorem 9.** *Let  $Z$  be a real algebraic variety and  $f$  a rational function on  $Z$  with continuous extension  $f^c$ . The following are equivalent.*

- (1)  $f^c$  is hereditarily rational.
- (2) For every real algebraic variety  $X$  that contains  $Z$  as a closed subvariety,  $f^c$  extends to a (continuous and) hereditarily rational function  $F$  on  $X$ .
- (3) For every real algebraic variety  $X$  that contains  $Z$  as a closed subvariety,  $f$  extends to a continuous rational function  $F$  on  $X$ .
- (4) Let  $X_0$  be a smooth real algebraic variety that contains  $Z$  as a closed subvariety. Then  $f$  extends to a continuous rational function  $F_0$  on  $X_0$ .

Proof. It is clear that (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) and (4)  $\Rightarrow$  (1) holds by Proposition 7. Thus we need to show that (1)  $\Rightarrow$  (2).

We can embed  $X$  into a smooth real algebraic variety  $X'$ . If we can extend  $f$  to  $X'$  then its restriction to  $X$  gives the required extension. Thus we may assume to start with that  $X$  is smooth (or even that  $X = \mathbb{R}^N$  for some  $N$ ).

We prove the following more precise version.

**Proposition 10.** *Let  $X$  be a smooth real algebraic variety and  $W \subset Z \subset X$  closed subvarieties. Let  $f$  be a continuous, hereditarily rational function on  $Z$  that is regular on  $Z \setminus W$ . Then  $f$  extends to a continuous, hereditarily rational function  $F$  on  $X$  that is regular on  $X \setminus W$ .*

Proof. We use induction on  $\dim Z$ . The case  $\dim Z = 0$  is obvious.

If  $W$  is replaced by a smaller set, the assertion gets stronger. Hence we may assume that  $W \subset Z$  is the smallest set such that  $f$  is regular on  $Z \setminus W$ . Since every rational function is regular on a Zariski dense open set,  $W$  is nowhere dense in  $Z$ . In particular,  $\dim W < \dim Z$ .

Since  $f$  is hereditarily rational,  $f^c|_W$  is also hereditarily rational. Thus, by induction, there is a hereditarily rational function  $F_1$  on  $X$  that is regular on  $X \setminus W$  and such that  $F_1^c|_W = f^c|_W$ . Set  $f_2 := f - F_1|_W$ . Then  $f_2^c$  vanishes on  $W$  and it is enough to show that  $f_2$  extends to a continuous, hereditarily rational function  $F_2$  on  $X$  that is regular on  $X \setminus W$ .

Let  $I_2$  be the ideal of those regular functions  $\phi$  on  $Z$  such that  $\phi f_2$  is regular. Let  $\phi_1, \dots, \phi_r$  be a set of generators of  $I_2$ . Set  $q := \phi_1^2 + \cdots + \phi_r^2$  then  $p := q f_2$  is regular and  $f_2 = p/q$ . Note further that  $W$  is the zero set of  $q$  and the zero set of  $p$  contains  $W$ .

A regular function on a closed subvariety  $Z$  of a real algebraic variety  $X$  always extends to a regular function on the whole variety. (This follows, for instance, from [BCR98, 4.4.5] or see Lemma 13 for a more general argument.)

Thus  $p$  (resp.  $q$ ) extend to regular functions  $P$  (resp.  $Q$ ) on  $X$ . Usually  $P/Q$  is not continuous but we will improve it in two steps.

Let  $H$  be a regular function on  $X$  whose zero set is  $Z$ . Then  $Q^2 + H^2$  vanishes only on  $W$  and its restriction to  $Z$  equals  $q^2$ . Thus

$$G := \frac{PQ}{Q^2 + H^2}$$

is a rational function on  $X$  that is regular on  $X \setminus W$  and whose restriction to  $Z \setminus W$  equals  $f_2$ . However, usually it is not continuous on  $X$  near  $W$  hence we need one more correction.

*Claim.* For  $n \gg 1$  the function

$$F_{2n} := G \cdot \frac{Q^{2n}}{Q^{2n} + H^2} = \frac{PQ}{Q^2 + H^2} \cdot \frac{Q^{2n}}{Q^{2n} + H^2}$$

is continuous on  $X$ .

It is clear that the restriction of  $F_{2n}$  to  $Z \setminus W$  equals  $f_2$ , thus, once the Claim is proved,  $F := F_1 + F_{2n}$  satisfies all the requirements.

In order to prove the claim, we work on the variety  $\pi : X_1 \rightarrow X$  obtained by blowing up the ideal  $(PQ, Q^2 + H^2)$ . Equivalently,  $X_1$  is the Zariski closure of the graph of  $G$  in  $X \times \mathbb{RP}^1$ .

Since  $W$  is the common zero set  $(PQ = Q^2 + H^2 = 0)$ , the exceptional set of  $\pi$  is  $E := \pi^{-1}(W) \cong W \times \mathbb{RP}^1$  and  $\pi$  is an isomorphism over  $X \setminus W$ . We prove that

$$(F_{2n}|_{X \setminus W}) \circ \pi$$

extends to a continuous function  $(F_{2n} \circ \pi)^c$  that vanishes on the exceptional set  $E$ . Thus  $(F_{2n} \circ \pi)^c$  descends to a continuous function on  $X$  that vanishes on  $W$ .

The existence of such a continuous extension is a local question on  $X_1$  and we use different arguments on different charts. A slight complication is that one of our charts is only semialgebraic.

Let  $Z^* \subset X_1$  be the Euclidean closure of  $\pi^{-1}(Z \setminus W)$ . Note that  $Z^*$  is a closed semialgebraic subset but it is usually not real algebraic.

On  $X_1$  one can identify the rational function  $G \circ \pi$  with the restriction of the second projection  $\pi_2 : W \times \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ .

On  $Z^*$  we thus have a continuous function  $f_2^c \circ \pi$  that vanishes on  $Z^* \cap E$  and a regular (hence continuous) map  $\pi_2|_{Z^*} : Z^* \rightarrow \mathbb{RP}^1$ . Using the natural identification  $\mathbb{R}^1 = \mathbb{RP}^1 \setminus \{\infty\}$ , these two agree on the open set  $\pi^{-1}(Z \setminus W)$ , hence they agree on  $Z^*$ . (Note that, as in Example 5, the two functions might not agree on the Zariski closure of  $\pi^{-1}(Z \setminus W)$ ; this is why we work with  $Z^*$ .) Thus  $\pi_2^{-1}(\infty)$  is disjoint from  $Z^*$ , and hence there is a Zariski open neighborhood  $U^* \subset X_1$  of  $Z^*$  such that  $\pi_2$  defines a regular function on  $U^*$ . This gives the extension of  $(G|_{X \setminus W}) \circ \pi$  to a regular function on  $U^*$  that vanishes along  $Z^* \cap E$ . (It is not clear that  $(G \circ \pi)|_{U^*}$  vanishes along  $U^* \cap E$ .)

Note further that

$$\frac{Q^{2n}}{Q^{2n} + H^2} \circ \pi$$

is a bounded regular function on  $X_1 \setminus E$ . Therefore the restriction of the product

$$(F_{2n} \circ \pi)|_{U^*} = (G \circ \pi) \cdot \left( \frac{Q^{2n}}{Q^{2n} + H^2} \circ \pi \right)|_{U^*}$$

extends to a function that vanishes and is continuous at every point of  $Z^* \cap E$ . However, we have not proved so far that it is defined along  $(U^* \cap E) \setminus Z^*$ .

The other chart is  $V^* := X_1 \setminus Z^*$ ; it is semialgebraic and Euclidean open in  $X_1$ . Here we write  $F_{2n} \circ \pi$  in the form

$$F_{2n} \circ \pi = (P \circ \pi) \cdot \left( \frac{Q^{2n-1}}{H^2} \circ \pi \right) \cdot \left( \frac{Q^2}{Q^2 + H^2} \circ \pi \right) \cdot \left( \frac{H^2}{Q^{2n} + H^2} \circ \pi \right).$$

The last 2 factors are bounded regular functions on  $V^* \setminus E$  and  $P$  is a regular function that vanishes along  $E \cap V^*$ .

Note that on  $V^*$  the function  $H \circ \pi$  vanishes only along  $E$  and  $Q \circ \pi$  also vanishes along  $E$ . We can thus apply Theorem 11 to conclude that  $(Q^{2n-1}/H^2) \circ \pi$  extends to a continuous (and semialgebraic) function on  $V^*$  for  $n \gg 1$ . Thus  $F_{2n} \circ \pi$  extends to a continuous function on  $V^*$  that vanishes along  $E \cap V^*$  for  $n \gg 1$ .

Putting the two charts together we conclude that  $F_{2n} \circ \pi$  extends to a continuous function on  $X_1$  that vanishes along  $E$ .  $\square$

We have used the following version of the Łojasiewicz inequality given in [BCR98, Thm.2.6.6], see also [Now11].

**Theorem 11** (Łojasiewicz Inequality). *Let  $V$  be a locally closed, semialgebraic subset of  $\mathbb{R}^N$  and  $\phi, \psi : V \rightarrow \mathbb{R}$  continuous semialgebraic functions. Assume that  $\{x \in V : \phi(x) = 0\} \subset \{x \in V : \psi(x) = 0\}$ . Then there exist a positive integer  $n$  and a continuous semialgebraic function  $\rho$  such that  $\psi^n = \rho\phi$ .*  $\square$

The following example shows that, even in very simple cases, the extension of continuous rational functions is not entirely trivial.

**Example 12.** Let  $X = \mathbb{R}^2$  and  $Z \subset \mathbb{R}^2$  the cuspidal cubic with equation  $(x^2 - y^3 = 0)$ . Consider the rational function  $f(x, y) := y^2/x$ .  $Z$  can be parametrized as  $x = t^3, y = t^2$  and then  $f(t^3, t^2) = t$  is clearly continuous.

First we claim that there are no regular functions  $P, Q$  such that  $P|_Z = y^2, Q|_Z = x$  and  $P/Q$  is continuous. In fact,  $P/Q$  can not even be bounded. Indeed, any such extension would be of the form

$$P = y^2 + P_1(x^2 - y^3) \quad \text{and} \quad Q = x + Q_1(x^2 - y^3)$$

where  $P_1, Q_1$  are regular. Thus

$$\frac{P}{Q}|_{y\text{-axis}} = \frac{y^2 - y^3 P_1}{-y^3 Q_1} = \frac{1 - y P_1}{-y Q_1}$$

has a pole at  $y = 0$ .

Our first two improvements are

$$\frac{y^2 x}{x^2 + (x^2 - y^3)^2} \quad \text{and} \quad \frac{y^2 x}{x^2 + (x^2 - y^3)^2} \cdot \frac{x^2}{x^4 + (x^2 - y^3)^2}.$$

For both of these, the limit along the curve  $(t^2, t)$  is 1, hence they are not continuous at the origin. The next improvement is

$$F_4 := \frac{y^2 x}{x^2 + (x^2 - y^3)^2} \cdot \frac{x^4}{x^4 + (x^2 - y^3)^2}.$$



This turns out to be continuous as can be seen either by two blow-ups or by the following direct computation. Performing the change of variables  $x = u^3$ ,  $y = u^2v$  we get

$$|F_4(u^3, u^2v)| = \left| \frac{uv^2}{(1 + u^6(1 - v^3)^2) \cdot (1 + (1 - v^3)^2)} \right| \leq |u| \cdot \frac{v^2}{1 + (1 - v^3)^2}.$$

The last factor is uniformly bounded by a constant  $C$  for  $v \in \mathbb{R}$ , thus we conclude that

$$|F_4(x, y)| \leq C|x|^{1/3}.$$

### 3. VARIETIES OVER OTHER TOPOLOGICAL FIELDS

Let  $K$  be any topological field. The  $K$ -points  $X(K)$  of any  $K$ -variety inherit from  $K$  a topology, called the  $K$ -topology. One can then consider rational functions  $f$  on  $X$  that are continuous on  $X(K)$ . This does not seem to be a very interesting notion in general, unless  $K$  satisfies the following *density property*.

(DP) If  $X$  is smooth, irreducible and  $\emptyset \neq U \subset X$  is Zariski open then  $U(K)$  is dense in  $X(K)$  in the  $K$ -topology.

Note that if (DP) holds for all smooth curves then it holds for all varieties. It is easy to see that if  $K$  is not discrete and the implicit function theorem holds over  $K$  then  $K$  has the above density property. Such examples are the  $p$ -adic fields  $\mathbb{Q}_p$ , their finite extensions and, more generally, quotient fields of complete local rings.

If  $K$  is algebraically closed, for instance  $K = \mathbb{C}$ , then every continuous rational function on a normal variety is regular, so we do not get a new notion. In general, the study of continuous rational functions leads to the concept of *seminormality* and *seminormalization*; see [AN67, AB69] or [Kol13, Sec.10.2] for a recent treatment.

The proofs of Section 2 all work over such fields  $K$ , as long as  $K$  is not algebraically closed. The only step that needs additional proof is the assertion that every regular function on a subvariety extends to the whole variety. This is proved by a slight modification of the usual arguments that apply when  $k$  is algebraically closed [Sha94, Sec.I.3.2] or real closed [BCR98, 3.2.3].

**Lemma 13** (Extending regular functions). *Let  $k$  be a field,  $X$  an affine  $k$ -variety and  $Z \subset X$  a closed subvariety. Let  $f$  be a rational function on  $Z$  that is regular at all points of  $Z(k)$ . Then there is a rational function  $F$  on  $X$  that is regular at all points of  $X(k)$  and such that  $F|_Z = f$ .*

*Proof.* If  $k$  is algebraically closed, then  $f$  is regular on  $Z$  hence it extends to a regular function on  $X$ .

If  $k$  is not algebraically closed, then, as a auxiliary step, we claim that there are homogeneous polynomials  $G(x_1, \dots, x_r)$  in any number of variables whose only zero on  $k^r$  is  $(0, \dots, 0)$ .

Indeed, if  $k$  is real, then we can take  $G = \sum x_i^2$ . By a theorem of Artin–Schreier, if  $k$  is not real closed, then it has finite extensions  $L/k$  whose degree is arbitrary large (cf. [Jac80, Sec.11.7]). If  $c_1, \dots, c_r$  is a  $k$ -basis of  $L$ , then  $G = \text{Norm}_{L/k}(\sum x_i c_i)$  works.

Now we construct the extension of  $f$  as follows.

For every  $z \in Z(k)$  we can write  $f = p_z/q_z$  where  $q_z(z) \neq 0$ . After multiplying both  $p_z, q_z$  with a suitable polynomial, we can assume that  $p_z, q_z$  are regular on  $Z$

and then extend them to regular functions on  $X$ . By assumption,  $\bigcap_{z \in Z(k)} (q_z = 0)$  is disjoint from  $Z(k)$ . Choose finitely many  $z_1, \dots, z_m \in Z(k)$  such that

$$\bigcap_{i=1}^m (q_{z_i} = 0) = \bigcap_{z \in Z(k)} (q_z = 0).$$

Let  $q_{m+1}, \dots, q_r$  be defining equations of  $Z \subset X$ . Set  $q_i := q_{z_i}$  and  $p_i := p_{z_i}$  for  $i = 1, \dots, m$  and  $p_i = q_i$  for  $i = m+1, \dots, r$ . Write (non-uniquely)  $G = \sum G_i x_i$  and finally set

$$F := \frac{\sum_{i=1}^r G_i(q_1, \dots, q_r) p_i}{G(q_1, \dots, q_r)}.$$

Since the  $q_i$  have no common zero on  $X(k)$ , we see that  $F$  is regular at all points of  $X(k)$ .

Along  $Z$ ,  $p_i = f q_i$  for  $i = 1, \dots, m$  by construction and for  $i = m+1, \dots, r$  since then both sides are 0. Thus

$$F|_Z := \frac{\sum_{i=1}^r G_i(q_1, \dots, q_r) f q_i}{G(q_1, \dots, q_r)}|_Z = f \cdot \frac{\sum_{i=1}^r G_i(q_1, \dots, q_r) q_i}{G(q_1, \dots, q_r)}|_Z = f. \quad \square$$

**Acknowledgments.** We thank Ch. Fefferman, M. Jarden, F. Mangolte and B. Totaro for useful comments and questions. Partial financial support for JK was provided by the NSF under grant number DMS-0758275. For KN, research was partially supported by the Polish Ministry of Science and Higher Education under grant number N N201 372336.

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